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Three Interesting Properties of Metallic Ratios

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Abstract: The study of Metallic ratios has profound impact and applications not just in mathematics but also in various other branches of Science and Technology. In this paper, I will introduce the Metallic Ratios formally and prove three interesting and new properties related to them. These results will add more value to already known results on Metallic Ratios.

Keywords: Convergence, Divergence, Alternating Series, Leibniz Test, Comparison Test

INTRODUCTION

The study of Metallic ratios has been made two millennia ago. One of the most famous real number, the Golden Ratio is a special case of Metallic Ratios. Several painters and architects had used Golden Ratio in their work. Golden ratio and few other metallic ratios were found abundantly in nature. In this paper, I will prove three elementary but new results regarding these fascinating class of numbers.

DEFINITIONS

Metallic Ratios are sequence of numbers defined recursively through the relation $M_{n+2} = nM_{n+1} + M_n \ (2.1)$.

In particular, the *n*th Metallic Ratio M_n is defined to be the positive root of the equation $x^2 - nx - 1 = 0 \ (2.2)$.

With this definition, we obtain $M_n = \frac{n+\sqrt{n^2+4}}{2} \ (2.3)$ and $-\frac{1}{M_n} = \frac{n-\sqrt{n^2+4}}{2} \ (2.4)$.

If we consider *n* = 1, 2, 3 respectively in (2.3) then the numbers obtained are defined as Golden, Silver and Bronze Ratios respectively. Thus, the golden ratio, silver ratio and bronze ratio are special cases form the first three terms of the most general class of numbers called Metallic Ratios.

By considering *n* = 1 in (2.3), the golden ratio is given by $M_1 = \frac{1+\sqrt{5}}{2} \ (2.5)$

By considering *n* = 2 in (2.3), the silver ratio is given by $M_2 = 1+\sqrt{2} \ (2.6)$

By considering *n* = 3 in (2.3), the bronze ratio is given by $M_3 = \frac{3+\sqrt{13}}{2} \ (2.7)$

LIMITING CASE OF METALLIC RATIOS

In this section, I will determine the convergence or divergence of the sequence of metallic ratios when divided by n^k for some integer *k*.

Theorem 1

If *k* is any integer, then $\frac{M_n}{n^k} \to 0, 1$ and diverges (3.1) as $n \to \infty$, according to the cases $k > 1, k = 1, k < 1$ respectively.

Proof: By (2.3), we know that $M_n = \frac{n+\sqrt{n^2+4}}{2}$.

Hence, $\frac{M_n}{n^k} = \frac{1}{2}\left(\frac{1}{n^{k-1}} + \sqrt{\frac{1}{n^{2(k-1)}} + \frac{4}{n^{2k}}}\right) \ (3.2)$

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If $k > 1$, then $\frac{1}{n^{k-1}}, \frac{1}{n^{2(k-1)}}, \frac{1}{n^{2k}} \to 0$ as $n \to \infty$. Hence as $n \to \infty$, $\frac{M_n}{n^k} \to 0$ (3.3).

If $k = 1$, then as $n \to \infty$, from (3.2) we see that $\frac{M_n}{n^k} = \frac{1}{2}\left(1+\sqrt{1+\frac{4}{n^2}}\right) \to 1$ (3.4)

If $k < 1$, then as $n \to \infty$, $\frac{1}{n^{k-1}}, \frac{1}{n^{2(k-1)}}$ diverges. Hence, from (3.2), diverges (3.5)

This completes the proof.

ALTERNATING SERIES CORRESPONDING TO METALLIC RATIOS

In this section, I will consider the alternating series of reciprocals of metallic ratios and discuss its convergence through the following theorem.

Theorem 2

The alternating series of reciprocals of metallic ratios converges.

That is, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{M_n}$ is convergent (4.1)

Proof: Using (2.4), we obtain

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{M_n} = \sum_{n=1}^{\infty} (-1)^n \times \left(-\frac{1}{M_n}\right)$$

$$= \sum_{n=1}^{\infty} (-1)^n \left(\frac{n-\sqrt{n^2+4}}{2}\right) = -\frac{1}{2}\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n^2+4}-n\right) \quad (4.2)$$

If we consider $a_n = \sqrt{n^2+4}-n$ then for $n \geq 1$ we notice that

$$a_n - a_{n-1} = \left(\sqrt{n^2+4}-n\right) - \left(\sqrt{(n-1)^2+4}-(n-1)\right) = \left(\sqrt{n^2+4}-\sqrt{(n-1)^2+4}\right) - 1 \quad (4.3)$$

Since $n \geq 1$ we have $n^2+4 \leq n^2+4n+4 = (n+2)^2$. Thus, $\sqrt{n^2+4} \leq n+2$ (4.4)

Similarly for $n \geq 1$ we have $(n-1)^2+4 = n^2-2n+5 \leq n^2+2n+1 \leq (n+1)^2$.

Thus, $\sqrt{(n-1)^2+4} \leq n+1$ (4.5)

From (4.4) and (4.5), we get $\sqrt{n^2+4}-\sqrt{(n-1)^2+4} \leq (n+2)-(n+1) = 1$ (4.6)

Using (4.6) in (4.3), we get $a_n \leq a_{n-1}$. Thus, $a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots$

Hence $\{a_n\}$ forms a non-increasing sequence of positive terms (4.7)

Moreover, as $n \to \infty$ we have $\sqrt{n^2+4} = n \times \sqrt{1+\frac{4}{n^2}} = n + \frac{2}{n} - \frac{2}{n^3} + \cdots$

Hence $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sqrt{n^2+4}-n\right) = \lim_{n\to\infty}\left(\frac{2}{n}-\frac{2}{n^3}+\cdots\right) = 0$ (4.8)

Thus using (4.7) and (4.8), by Leibniz's test the alternating series $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n^2+4}-n\right)$ converges.

Hence from (4.2), we see that the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{M_n}$ is convergent.

This completes the proof.

SERIES OF RECIPROCALS OF METALLIC RATIOS

In this section, I will consider the series of reciprocals of metallic ratios. The following theorem addresses the convergence of such series.

Theorem 3

The series of reciprocals of metallic ratios diverges. That is, $\sum_{n=1}^{\infty}\frac{1}{M_n}$ is divergent (5.1)

Proof: From (2.4), we get $\sum_{n=1}^{\infty}\frac{1}{M_n}=\sum_{n=1}^{\infty}\frac{\sqrt{n^2+4}-n}{2}=\sum_{n=1}^{\infty}\frac{2}{\sqrt{n^2+4}+n}$ (5.2)

We now have $(n+2)^2-(n^2+4)=4n\geq 0$ for all $n\geq 1$. Thus, $\sqrt{n^2+4}\leq n+2$ (5.3)

Therefore, $n+\sqrt{n^2+4}\leq 2n+2=2(n+1)$ giving $\frac{2}{\sqrt{n^2+4}+n}\geq\frac{1}{n+1}$ (5.4)

Thus from (5.2) and (5.4), we have $\sum_{n=1}^{\infty}\frac{1}{M_n}=\sum_{n=1}^{\infty}\frac{2}{\sqrt{n^2+4}+n}\geq\sum_{n=1}^{\infty}\frac{1}{n+1}$ (5.5)

Since $\sum_{n=1}^{\infty}\frac{1}{n+1}$ is the Harmonic series without the first term 1, and since Harmonic series diverges, it follows that $\sum_{n=1}^{\infty}\frac{1}{n+1}$ diverges. Hence by comparison test, and using (5.5), we observe that the series $\sum_{n=1}^{\infty}\frac{1}{M_n}$ also diverges. This completes the proof.

CONCLUSION

Three new results related to Metallic Ratios have been established in this paper. The first result in equation (3.1) of theorem 1 addresses the asymptotic behavior of sequence of metallic ratios. In particular, it conveys the fact that the *n*th Metallic Ratio M_n is of order *n*. The second result in equation (4.1) of theorem 2 conveys the fact that the alternating series of reciprocals of metallic ratios is convergent. The third result established in equation (5.1) of theorem 3, proves that the series of reciprocals of metallic ratios is divergent. Thus, in view of theorems 2 and 3, we see that the series of reciprocals of metallic ratios is conditionally convergent but not absolutely convergent.

Since by theorem 1, we notice that the *n*th metallic ratio M_n is of order *n*, the series of reciprocals of metallic ratios behaves in the same way as Harmonic series (the series of reciprocals of natural numbers), in the sense that, the Harmonic series is conditionally convergent but not absolutely convergent. These three new results will add more value to existing ideas related to magnificent metallic ratios.

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